ON SOME MEAN VALUE RESULTS FOR THE ZETA-FUNCTION AND A DIVISOR PROBLEM II

Aleksandar Ivić and Wenguang Zhai[1]

Abstract. Let $d(n)$ be the number of divisors of n, let

$$\Delta(x) := \sum_{n\leqslant x} d(n) - x(\log x + 2\gamma - 1)$$

denote the error term in the classical Dirichlet divisor problem, and let $\zeta(s)$ denote the Riemann zeta-function. It is shown that

$$\int_0^T \Delta(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \ll T(\log T)^4.$$

Further, if $2 \leqslant k \leqslant 8$ is a fixed integer, then we prove the asymptotic formula

$$\int_1^T \Delta^k(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t = c_1(k)T^{1+\frac{k}{4}}\log T + c_2(k)T^{1+\frac{k}{4}} + O_\varepsilon(T^{1+\frac{k}{4}-\eta_k+\varepsilon}),$$

where $c_1(k)$ and $c_2(k)$ are explicit constants, and where

$$\eta_2 = 3/20, \eta_3 = \eta_4 = 1/10,\ \eta_5 = 3/80,\ \eta_6 = 35/4742,\ \eta_7 = 17/6312,\ \eta_8 = 8/9433.$$

The results depend on the power moments of $\Delta(t)$ and $E(T)$, the classical error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2}+it)|$.

[1]W. Zhai's work is supported by the National Key Basic Research Program of China (Grant No. 2013CB834201), the National Natural Science Foundation of China (Grant No. 11171344), and the Fundamental Research Funds for the Central Universities in China (2012YS01).

1991 *Mathematics Subject Classification*. 11M06, 11N37.

Key words and phrases. Dirichlet divisor problem, Riemann zeta-function, integral of the error term, mean value estimates.

1. Introduction

As usual, let

$$\Delta(x) \ := \sum_{n\leqslant x} d(n) - x(\log x + 2\gamma - 1) \qquad (x \geqslant 2) \tag{1.1}$$

denote the error term in the classical Dirichlet divisor problem (see e.g., Chapter 3 of [4]). Also let

$$E(T) \ := \int_0^T |\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t - T\Big(\log\big(\frac{T}{2\pi}\big) + 2\gamma - 1\Big) \qquad (T \geqslant 2) \tag{1.2}$$

denote the error term in the mean square formula for $|\zeta(\frac12 + it)|$. Here $d(n)$ is the number of all positive divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. In the first part of this work [9], the first author proved several results involving the mean values of $\Delta(x), E(t)$ and

$$\begin{aligned}\Delta^*(x) :&= -\Delta(x) + 2\Delta(2x) - \tfrac12\Delta(4x)\\ &= \tfrac12 \sum_{n\leqslant 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1),\end{aligned} \tag{1.3}$$

which is the "modified" divisor function, introduced and studied by M. Jutila [12], [13]. In view of F.V. Atkinson's classical explicit formula [1] for $E(T)$, which shows analogies between $\Delta(x)$ and $E(T)$, it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$ itself. Namely, M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) \ := \ E(t) - 2\pi\Delta^*\big(\frac{t}{2\pi}\big), \tag{1.4}$$

and in particular he proved that

$$\int_T^{T+H} (E^*(t))^2 \,\mathrm{d}t \ \ll_\varepsilon \ HT^{1/3}\log^3 T + T^{1+\varepsilon} \qquad (1 \leqslant H \leqslant T). \tag{1.5}$$

Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a(x) \ll_\varepsilon b(x)$ (same as $a(x) = O_\varepsilon(b(x))$) means that the $|a(x)| \leqslant Cb(x)$ for some $C = C(\varepsilon) > 0, x \geqslant x_0$. The significance of (1.5) is that, in view of (see e.g., [4, Chapter 15])

$$\int_0^T (\Delta^*(t))^2 \,\mathrm{d}t \ \sim \ AT^{3/2}, \quad \int_0^T E^2(t) \,\mathrm{d}t \ \sim \ BT^{3/2} \quad (A, B > 0,\ T \to \infty), \tag{1.6}$$

it transpires that $E^*(t)$ is in the mean square sense of a lower order of magnitude than either $\Delta^*(t)$ or $E(t)$. A similar mean square formula holds for $\Delta(t)$ as well, and actually sharper formulas are known in all three cases; for this see the paper of Lau–Tsang [15]. We also refer the reader to the review paper [21] of K.-M. Tsang on this subject.

Thus it seemed interesting to study the interplay between $\Delta^*(t)$ (and $\Delta(t)$) and $\zeta(s)$. Mean values (or moments) of $|\zeta(\frac12+it)|$ represent one of the central themes in the theory of $\zeta(s)$, and they have been studied extensively. There are two monographs dedicated solely to them: the author's [7], and that of K. Ramachandra [18]. In [9] it was proved that, for $T^{2/3+\varepsilon} \leqslant H = H(T) \leqslant T$, we have

$$\int_T^{T+H} \Delta^*\Bigl(\frac{t}{2\pi}\Bigr)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \ll HT^{1/6}\log^{7/2}T. \tag{1.7}$$

It was also proved that if C is a suitable positive constant, then

$$\int_0^T \Bigl(\Delta^*\Bigl(\frac{t}{2\pi}\Bigr)\Bigr)^2|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \frac{C}{4\pi^2}T^{3/2}\Bigl(\log\frac{T}{2\pi}+2\gamma-\frac23\Bigr)+O_\varepsilon(T^{17/12+\varepsilon}), \tag{1.8}$$

and if D is another suitable positive constant, then

$$\int_0^T \Bigl(\Delta^*\Bigl(\frac{t}{2\pi}\Bigr)\Bigr)^3|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = DT^{7/4}\Bigl(\log\frac{T}{2\pi}+2\gamma-\frac47\Bigr)+O_\varepsilon(T^{27/16+\varepsilon}). \tag{1.9}$$

The proofs of (1.8) and (1.9), given in [9], exploited the special structure of $\Delta^*\bigl(\frac{t}{2\pi}\bigr)$ and could not be readily extended to deal with $\Delta^*(\alpha t)$ or $\Delta(\alpha t)$ for a given $\alpha > 0$.

2. Statement of results

This paper is a continuation of the first author's paper [9] and the second author's papers [24], [25], where he investigated the high-power moments of $\Delta(x)$ and $E(t)$.

Namely it is conjectured that the asymptotic formula

$$\int_0^T \Delta^k(t)\,\mathrm{d}t = C_kT^{1+k/4}+O_\varepsilon(T^{1+k/4-c(k)+\varepsilon}) \tag{2.1}$$

holds with an explicit constant C_k and some $c(k) > 0$, when $k > 1$ is a given natural number. An asymptotic formula analogous to (2.1) is also conjectured for the moments of $E(t)$. The case $k = 2$ (the mean square) of (2.1) is classic, and it is now known that (see Lau–Tsang [15])

$$\int_0^T \Delta^2(t)\,\mathrm{d}t = C_2T^{3/2}+O(T\log^3T\log\log T), \tag{2.2}$$

with

$$C_2 = (6\pi^2)^{-1}\sum_{n=1}^\infty d^2(n)n^{-3/2} = (6\pi^2)^{-1}\zeta^4(3/2)/\zeta(3) = 0.25045\ldots,$$

and a formula analogous to (2.2) holds for the mean square of $E(t)$. A detailed discussion concerning the integral in (2.1) in the general case is to be found in the second author's paper [24], Part II, where (2.1) is established for $5 \leqslant k \leqslant 9$, with explicit values of $c(k)$. For $k = 3$ the best known value is $c(3) = 7/20$ (Ivić–Sargos [11]) and for $k = 4$ it is $c(4) = 3/28$ (W. Zhai [24]), and K.-L. Kong [14] has just obtained $c(4) = 1/8$. Like in the problem of the evaluation of the moments $\int_0^T |\zeta(\frac12 + it)|^k\,\mathrm{d}t$ and similar problems, the problem becomes progressively more difficult as k increases. It is curious that, for $2 \leqslant k \leqslant 9$, when it is as present known that the asymptotic formula (2.1) holds, all the constants C_k are positive for odd k, implying that the values of $\Delta(t)$ are more biased towards positive values. Whether this phenomenon will also happen for odd $k > 9$, should (2.1) continue to hold, is unclear.

In this paper we are interested in a similar, but more involved problem, namely the asymptotic evaluation of the integrals of $\Delta^k(t)|\zeta(\frac12+it)|^2$ when $k \in \mathbb{N}$ is fixed. We succeeded in applying the existing results on the moments of $\Delta(t)$ and $E(t)$ to the evaluation of the integrals of $\Delta^k(t)|\zeta(\frac12 + it)|^2$. Our methods at present work for $1 \leqslant k \leqslant 8$, and the results are as follows.

THEOREM 1. *We have*

$$\int_0^T \Delta(t)|\zeta(\tfrac12 + it)|^2\,\mathrm{d}t \ll T(\log T)^4. \tag{2.3}$$

THEOREM 2. *If k is a fixed integer for which $2 \leqslant k \leqslant 8$, then we have*

$$\int_1^T \Delta^k(t)|\zeta(\tfrac12 + it)|^2\,\mathrm{d}t = c_1(k)T^{1+\frac{k}{4}}\log T + c_2(k)T^{1+\frac{k}{4}} + O_\varepsilon(T^{1+\frac{k}{4}-\eta_k+\varepsilon}), \tag{2.4}$$

where $c_1(k)$ and $c_2(k)$ are explicit constants, and where

$$\eta_2 = \eta_3 = \eta_4 = 1/10,\ \eta_5 = 3/80,\ \eta_6 = 35/4742,\ \eta_7 = 17/6312,\ \eta_8 = 8/9433.$$

Note that the values of η_2, η_3, η_4 in Theorem 2 are identical, which is due to the general argument used in the proof in Section 5. However, we can combine the arguments of Theorem 1 and Theorem 2 to obtain improvements on the values of η_2 and η_3. We shall give the details only for η_2, while the case of η_3 is technically quite complicated. The result is

THEOREM 3. *When* $k=2$*, we can take* $\eta_2 = 3/20$ *in Theorem* 2.

Remark 1. Note that, for $H=T$, (2.3) improves (1.7) a lot. It is an open problem to find the lower bound for the integral in (2.3), since it is well-known that $\Delta(x)$ changes sign in every interval of the form $[T, T+A\sqrt{T}]$ for a suitable $A>0$ and $T \geqslant T_0$ (see the first author's paper [6]). On the other hand, one has (by (3.1) and (3.7) of [4]) that

$$\int_1^X \Delta(x)\,\mathrm{d}x = \tfrac14 X + O(X^{3/4}).$$

Using this formula it may be conjectured that

$$\int_1^T \Delta(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \frac{T}{4}\Bigl(\log\frac{T}{2\pi} + 2\gamma - 1\Bigr) + O_\varepsilon(T^{3/4+\varepsilon}), \tag{2.5}$$

however obtaining any asymptotic formula for the integral in (2.5) is difficult.

Corollary 1. We also have

$$\int_0^T E^*(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \ \ll\ T(\log T)^4. \tag{2.6}$$

This follows from (1.4), (2.1) (since it will hold with $\Delta^*(t/(2\pi))$ instead od $\Delta(t)$),

$$\int_0^T E(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \pi T\Bigl(\log\frac{T}{2\pi} + 2\gamma - 1\Bigr) + U(T), \tag{2.7}$$

where

$$U(T) = O(T^{3/4}\log T), \quad U(T) = \Omega_\pm(T^{3/4}\log T). \tag{2.8}$$

The asymptotic formulas (2.7)–(2.8) are due to the first author [5]. They show, up to the numerical constants which are involved, the true order of magnitude of the function $U(T)$. Here the symbol $f(x) = \Omega_\pm(g(x))$ has its standard meaning, namely that both $\limsup\limits_{x\to\infty} f(x)/g(x) > 0$ and $\liminf\limits_{x\to\infty} f(x)/g(x) < 0$ holds.

The analogy between (2.5) and (2.7) is obvious, however the latter is much less difficult. Namely the defining relation (1.2) yields, by differentiation,

$$|\zeta(\tfrac12+it)|^2 = \log\Bigl(\frac{t}{2\pi}\Bigr) + 2\gamma + E'(t), \tag{2.10}$$

and one can easily integrate $E^k(t)E'(t)$ $(k\in\mathbb{N})$. Thus the integral in (2.4) is more difficult to evaluate than the corresponding problem when $\Delta^k(t)$ is replaced by $E^k(t)$.

Remark 2. The methods of proof of (2.4) allow one to carry over the results of Theorem 1, Theorem 2 and Theorem 3 to integrals where $\Delta(t)$ is replaced by $\Delta(\alpha t)$ or $\Delta^*(\alpha t)$ for any given $\alpha > 0$.

Remark 3. It would be interesting to analyze the error term in (2.4) and see how small it can be, i.e., to obtain an omega-result (recall that $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$).

Remark 4. For $k = 2$ one can compare (2.4) with the corresponding result of the first author [5], where it was obtained that

$$\int_0^T E^2(t)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t = D_2 T^{3/2}\Bigl(\log\frac{T}{2\pi} + 2\gamma - \frac{2}{3}\Bigr) + O(T\log^6 T),$$

where

$$D_2 \;=\; \frac{2\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)}.$$

Remark 5. Finally we indicate two possible generalizations of our results. Let, as usual, $r(n) = \sum_{n=a^2+b^2} 1$ denote the number of ways n may be represented as a sum of two integer squares, and let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, and denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$. We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$. The classical example is $a(n) = \tau(n)$ ($\kappa = 12$), the Ramanujan τ-function defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n \;=\; x\bigl\{(1-x)(1-x^2)(1-x^3)\cdots\bigr\}^{24} \qquad (\,|x| < 1).$$

If $P(x) := \sum_{n\leqslant x} r(n) - \pi x$ denotes then the error term in the classical circle problem and $A(x) := \sum_{n\leqslant x} a(n)$, then Theorem 2 can be generalized to integrals

$$\int_0^T P^k(t)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t, \quad \int_0^T A^k(t)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t, \tag{2.11}$$

more precisely if $A(t)$ replaced by the normalized function $A^*(t) := \sum_{n\leqslant t} a(n)n^{\frac{1-\kappa}{2}}$, since $a(n)$ behaves similarly to $n^{(\kappa-1)/2}d(n)$. For the analogues of Lemma 4 to $P(x)$ and $A(x)$ the reader should see e.g., section 3 of [8]. The analogue of (3.1) for $\Delta(x)$ will hold with a poorer θ (with $\theta = 1/3$ in case of $A^*(x)$), and the analogues of the exponents η_k will not be as good as those of Theorem 2.

3. The necessary lemmas

In this section we shall state some lemmas needed for the proof of our theorems. The proofs of the theorems themselves will be given in Section 4, Section 5 and Section 6.

LEMMA 1. *There exists a constant θ such that $1/4 \leqslant \theta < 1/3$ and*

$$\Delta(x) \ll_\varepsilon x^{\theta+\varepsilon}, \quad E(t) \ll_\varepsilon t^{\theta+\varepsilon}. \tag{3.1}$$

In particular, we can take $\theta = 131/416 = 0.3149\cdots$.

The proofs of the bounds in (3.1) are due to M.N. Huxley [3] and N. Watt [23], respectively, and they are the sharpest ones known. It is commonly conjectured that $\theta = 1/4$ is permissible, but this is out of reach at present. It is known that $\theta < 1/4$ cannot hold (see e.g., [4], Chapter 13 and Chapter 15).

LEMMA 2. *Suppose θ is the constant in Lemma 1. Then for any A satisfying $0 \leqslant A \leqslant 11$ we have*

$$\int_1^T |\Delta(x)|^A \,\mathrm{d}x \ll_\varepsilon T^{1+M(A)+\varepsilon} \tag{3.2}$$

and

$$\int_1^T |E(t)|^A \,\mathrm{d}t \ll_\varepsilon T^{1+M(A)+\varepsilon}, \tag{3.3}$$

where

$$M(A) := \max\left(\frac{A}{4}, \theta(A-2)\right). \tag{3.4}$$

We note that, for real $k \in [0,9]$, the limits

$$E_k \; := \lim_{T\to\infty} T^{-1-k/4} \int_0^T |E(t)|^k \,\mathrm{d}t \tag{3.5}$$

exist. The analogous result holds also for the moments of $\Delta(t)$. This was proved by D.R. Heath-Brown [2], who used (3.4) in his proof. He also showed that the limits of moments (both of $\Delta(t)$ and $E(t)$) without absolute values also exist when $k = 1,3,5,7$ or 9. The merit of (3.5) that it gets rid of "ε" and establishes the existence of the limit (but without an error term).

LEMMA 3. *We have*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t = TQ_4(\log T) + O(T^{2/3}\log^8 T), \tag{3.6}$$

where $Q_4(x)$ is an explicit polynomial of degree four in x with leading coefficient $1/(2\pi^2)$.

This result was proved first (with error term $O(T^{2/3}\log^C T)$) by Y. Motohashi and the author [10]. The value $C = 8$ was given by Y. Motohashi in his monograph [17]. We shall not need the full strength of (3.6), but just the upper bound $O(T\log^4 T)$ for the integral in question.

LEMMA 4. *For $1 \leqslant N \ll x$ we have*

$$\Delta(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\leqslant N} d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}). \tag{3.7}$$

and

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\leqslant N} (-1)^n d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}). \tag{3.8}$$

The expression (3.8) for $\Delta^*(x)$ (see [4], Chapter 15) is the analogue of the classical truncated Voronoï formula (3.7) for $\Delta(x)$ (ibid. Chapter 3), only the sum in the expression for $\Delta^*(x)$ has an additional factor $(-1)^n$. Actually G.F. Voronoï [22] proved long ago an explicit formula for $\Delta(x)$ as a series containing the Bessel functions K_1 and Y_1 (see e.g., [4], Chapter 3). However, to avoid the questions of convergence it is in practice usually more expedient to work with (3.7), which is sufficient for many purposes.

LEMMA 5. *For $Q \gg x \gg 1$ we have*

$$\Delta(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\leqslant Q} d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + F(x), \tag{3.9}$$

where $F(x) \ll x^{-1/4}$ if $||x|| \gg x^{5/2}Q^{-1/2}$, and we always have $F(x) \ll_\varepsilon x^\varepsilon$.

This result ($||x||$ denotes as usual the distance of x to the nearest integer) is due to T. Meurman [16]. It shows that, unless x is close to an integer, the error term in the truncated Voronoï formula for $\Delta(x)$ is small.

LEMMA 6. *Let $k \geqslant 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers n_1, n_2, n_3, n_4 such that $N < n_1, n_2, n_3, n_4 \leqslant 2N$ and*

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| \;<\; \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon \; N^\varepsilon(N^4\delta + N^2). \tag{3.10}$$

Lemma 6 was proved by analytic methods by Robert–Sargos [19]. When $k = 2$, it represents a powerful arithmetic tools which is essential in the analysis when the biquadrate of exponential sums involving $\sqrt{n}$ appears.

LEMMA 7. *We have*

$$\sum_{n\leqslant x} d^2(n) \;=\; \frac{1}{\pi^2}x\log^3 x + O(x\log^2 x). \tag{3.11}$$

This is a well-known elementary formula; see e.g., page 141 of [4]. It follows from the series representation

$$\sum_{n=1}^{\infty} d^2(n)n^{-s} = \frac{\zeta^4(s)}{\zeta(2s)} \qquad (\Re s > 1).$$

LEMMA 8. *For* $1 \leqslant r \ll x$ *we have*

$$\sum_{n\leqslant x} d(n)d(n+r) \ll \sum_{d|r}\frac{1}{d}\cdot x\log^2 x.$$

This follows e.g., from a theorem of P. Shiu [20] on multiplicative functions.

LEMMA 9 . *Let* $0 < A < A'$ *be any two fixed constants such that* $AT < N < A'T$, *and let* $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. *Then*

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T),$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4}\sum_{n\leqslant N}(-1)^n d(n)n^{-3/4}e(T,n)\cos(f(T,n)),$$

$$\Sigma_2(T) = -2\sum_{n\leqslant N'} d(n)n^{-1/2}\Bigl(\log\frac{T}{2\pi n}\Bigr)^{-1}\cos\left(T\log\Bigl(\frac{T}{2\pi n}\Bigr) - T + \tfrac{1}{4}\pi\right),$$

with

$$\begin{aligned} f(T,n) &= 2T\operatorname{arsinh}\Bigl(\sqrt{\frac{\pi n}{2T}}\Bigr) + \sqrt{2\pi nT + \pi^2n^2} - \tfrac{1}{4}\pi \\ &= -\tfrac{1}{4}\pi + 2\sqrt{2\pi nT} + \tfrac{1}{6}\sqrt{2\pi^3}n^{3/2}T^{-1/2} + a_5n^{5/2}T^{-3/2} + a_7n^{7/2}T^{-5/2} + \dots, \end{aligned}$$

$$
\begin{aligned}
e(T,n) &= (1+\pi n/(2T))^{-1/4}\Big\{(2T/\pi n)^{1/2}\mathrm{arsinh}\Big(\sqrt{\frac{\pi n}{2T}}\,\Big)\Big\}^{-1}\\
&= 1 + O(n/T) \qquad (1 \leqslant n < T),
\end{aligned}
$$

and $\operatorname{ar\,sinh} x = \log(x + \sqrt{1+x^2}\,)$.

This is the famous formula of F.V. Atkinson [1]; proofs can be also found in [4] and [7].

LEMMA 10. *Let* $p_1, p_2, \ldots, p_r > 0$ *and* $f_1(x), f_2(x), \ldots, f_r(x) \geqslant 0$ *be continuous functions in* $[a,b]$ $(a < b)$. *Then if*

$$
\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} = 1,
$$

we have

$$
\int_a^b f_1(x)\ldots f_r(x)\,\mathrm{d}x \leqslant \left(\int_a^b f_1^{p_1}(x)\,\mathrm{d}x\right)^{1/p_1} \cdots \left(\int_a^b f_r^{p_r}(x)\,\mathrm{d}x\right)^{1/p_r}.
$$

This is the classical Hölder inequality for integrals, the case $r = 2, p_1 = p_2 = 1/2$ being the Cauchy-Schwarz inequality. It will be repeatedly used in the proofs.

4. Proof of Theorem 1

It suffices to consider in (2.3) the integral over $[T,\, 2T]$, to replace then T by $T2^{-j}$ $(j = 1, 2, \ldots)$ and sum the resulting estimates. We suppose $T \leqslant t \leqslant 2T$, take $Q = T^7$ in Lemma 5 and write

$$
\Delta(t) = \Delta_1(t) + \Delta_2(t) + F(t), \tag{4.1}
$$

where $F(t)$ is as in Lemma 5, and

$$
\begin{aligned}
\Delta_1(t) &:= \frac{t^{1/4}}{\pi\sqrt{2}} \sum_{n \leqslant T} d(n) n^{-3/4} \cos(4\pi\sqrt{nt} - \tfrac{\pi}{4}),\\
\Delta_2(t) &:= \frac{t^{1/4}}{\pi\sqrt{2}} \sum_{T < n \leqslant Q} d(n) n^{-3/4} \cos(4\pi\sqrt{nt} - \tfrac{\pi}{4}).
\end{aligned} \tag{4.2}
$$

Therefore

$$
\int_T^{2T} \Delta(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t = \int_T^{2T} (\Delta_1(t) + \Delta_2(t) + F(t))|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t.
$$

By the Cauchy-Schwarz inequality (Lemma 10) it is seen that the term $F(t)$ in (4.1) makes a contribution of $O(T^{3/4}\log T)$. The contribution containing $\Delta_2(t)$ is, by the first derivative test (see e.g., Lemma 2.1 of [4]), Lemma 3 and Lemma 7, (4.3)

$$
\begin{aligned}
&\ll T^{1/4}\Big\{\int_T^{2T}\Big|\sum_{T<n\leqslant Q} d(n)n^{-3/4}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\Big|^2\,\mathrm{d}t\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\Big\}^{\frac{1}{2}}\\
&\ll T^{3/4}\log^2 T\left\{T^{1/2}\log^3 T+T^{1/2}\sum_{T<m\neq n\leqslant Q}\frac{d(m)d(n)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

In the double sum in in (4.3), the contribution when $m \geqslant 4n$ or $n \geqslant 4m$ is $\ll \log^3 T$. The contribution of the remaining terms is, supposing $m > n$, setting $m = n + r$ and using Lemma 8,

$$
\ll \sum_{r\ll Q}\frac{1}{r}\sum_{n\leqslant Q}\frac{d(n)d(n+r)}{n} \ll \sum_{r\ll Q}\frac{1}{r}\sum_{d|r}\frac{1}{d}\log^3 T \ll \log^4 T.
$$

Therefore the contribution containing $\Delta_2(t)$ is

$$
\ll T^{3/4}\log^2 T(T^{1/2}\log^4 T)^{1/2} = T\log^4 T.
$$

Further we have, by (2.10),

$$
\begin{aligned}
&\int_T^{2T}\Delta_1(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\\
&=\int_T^{2T}\frac{t^{1/4}}{\pi\sqrt{2}}\sum_{n\leqslant T} d(n)n^{-3/4}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\Big(\log\tfrac{t}{2\pi}+2\gamma+E'(t)\Big)\,\mathrm{d}t \qquad (4.4)\\
&= I_1(T)+I_2(T),
\end{aligned}
$$

say, where by the first derivative test

$$
\begin{aligned}
I_1(T) :&= \int_T^{2T}\frac{t^{1/4}}{\pi\sqrt{2}}\sum_{n\leqslant T} d(n)n^{-3/4}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\Big(\log\tfrac{t}{2\pi}+2\gamma\Big)\,\mathrm{d}t\\
&\ll T^{1/4}\log T\cdot\sum_{n\leqslant T} d(n)n^{-3/4}T^{1/2}n^{-1/2}\\
&\ll T^{3/4}\log T,
\end{aligned}
$$

since $\sum_{n\geqslant 1} d(n)n^{-\alpha}$ converges for $\alpha > 1$. The integral $I_2(T)$, namely

$$
I_2(T) := \int_T^{2T} E'(t)\frac{t^{1/4}}{\pi\sqrt{2}}\sum_{n\leqslant T} d(n)n^{-3/4}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t,
$$

is integrated by parts. Since $E(t) \ll t^{1/3}$ (see e.g., Chapter 15 of [4], also follows trivially from Lemma 1), the integrated terms are trivially

$$\ll\ T^{\frac{1}{3}+\frac{1}{4}}T^{\frac{1}{4}}\log T\ \ll\ T^{\frac{5}{6}}\log T.$$

There remains a multiple of

$$
\begin{aligned}
&-\frac{1}{4}\int_T^{2T} t^{-3/4}E(t)\sum_{n\leqslant T} d(n)n^{-3/4}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t\\
&+2\pi\int_T^{2T} t^{-1/4}E(t)\sum_{n\leqslant T} d(n)n^{-1/4}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t.
\end{aligned}
\tag{4.5}
$$

Both integrals in (4.5) are estimated analogously, and clearly it is the latter which is larger. We replace $E(t)$ by the expression given by Atkinson's formula (see Lemma 9). Thus, taking $N = T$ in Atkinson's formula,

$$\int_T^{2T} t^{-1/4}E(t)\sum_{n\leqslant T} d(n)n^{-1/4}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t = J_1(T)+J_2(T)+J_3(T),$$

say, where

$$
\begin{aligned}
J_1(T) &:= \int_T^{2T} t^{-1/4}\textstyle\sum_1(t)\displaystyle\sum_{n\leqslant T} d(n)n^{-1/4}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t,\\
J_2(T) &:= \int_T^{2T} t^{-1/4}\textstyle\sum_2(t)\displaystyle\sum_{n\leqslant T} d(n)n^{-1/4}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t,\\
J_3(T) &:= \int_T^{2T} t^{-1/4}O(\log^2 T)\sum_{n\leqslant T} d(n)n^{-1/4}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t.
\end{aligned}
$$

The Cauchy-Schwarz inequality gives

$$
\begin{aligned}
&J_2(T)+J_3(T)\\
&\ll T^{-1/4}\left\{\int_T^{2T}\left(\textstyle\sum_2^2(t)+\log^4 T\right)\mathrm{d}t\int_T^{2T}\Big|\displaystyle\sum_{n\leqslant T} d(n)n^{-1/4}e^{4\pi i\sqrt{nt}}\Big|^2\mathrm{d}t\right\}^{1/2}.
\end{aligned}
\tag{4.6}
$$

But (see Chapter 15 of [4])

$$\int_T^{2T}\textstyle\sum_2^2(t)\,\mathrm{d}t\ \ll\ T\log^4 T,$$

since $\sum_2(t)$ is essentially a Dirichlet polynomial of length $\asymp T$. In the other integral in (4.6) we square out the sum and integrate. The contribution is

$$\begin{aligned}&\ll T\sum_{n\leqslant T} d^2(n)n^{-1/2} + \sum_{m\neq n\leqslant T}\frac{d(m)d(n)}{(mn)^{1/4}}\int_T^{2T} e^{4\pi i(\sqrt{m}-\sqrt{n})\sqrt{t}}\,\mathrm{d}t\\ &\ll T^{3/2}\log^3 T + T^{1/2}\sum_{m\neq n\leqslant T}\frac{d(m)d(n)}{(mn)^{1/4}|\sqrt{m}-\sqrt{n}|},\end{aligned}$$

by the first derivative test and Lemma 7. Note that if $m\leqslant n/2$, then $|\sqrt{m}-\sqrt{n}|^{-1}\ll n^{-1/2}$, while if $m>2n$, then $|\sqrt{m}-\sqrt{n}|^{-1}\ll m^{-1/2}$. When $m\asymp n$ the contribution is estimated, as in (4.3), by Lemma 6. In this way it is seen that

$$\int_T^{2T}\Bigl|\sum_{n\leqslant T} d(n)n^{-1/4}e^{4\pi i\sqrt{nt}}\Bigr|^2\,\mathrm{d}t \;\ll\; T^{3/2}\log^3 T, \tag{4.7}$$

and we obtain

$$J_2(T)+J_3(T)\ll T^{-1/4}T^{1/2}\log^2 T\cdot T^{3/4}\log^{3/2}T\ll T(\log T)^{7/2}.$$

It remains to deal with (c is a constant)

$$J_1(T) = c\int_T^{2T}\sum_{m\leqslant T}\frac{(-1)^m d(m)}{m^{3/4}}e(t,m)\cos f(t,m)\sum_{n\leqslant T}\frac{d(n)}{n^{1/4}}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4})\,\mathrm{d}t.$$

We split the sums over m,n into $O(\log^2 T)$ subsums with the ranges of summation $M<m\leqslant M'\leqslant 2M, N<n\leqslant N'\leqslant 2N$, respectively. We write the cosines as exponentials and then obtain $\ll\log^2 T$ sums of the form

$$\begin{aligned}&\sum_{M<m\leqslant M'}\frac{(-1)^m d(m)}{m^{3/4}}\sum_{N<n\leqslant N'}\frac{d(n)}{n^{1/4}}\times\\ &\times\int_T^{2T} e(t,m)\exp\Bigl(4\pi i\sqrt{nt}-i\sqrt{8\pi mt}-ia_3m^{3/2}t^{-1/2}-\dots\Bigr)\,\mathrm{d}t.\end{aligned}\tag{4.8}$$

There is also the expression with $+$ in place of $-$ in the exponential, and their conjugates, but it is (4.8) that is the relevant sum. The smooth function $e(t,m)$ ($=1+O(m/T)$) may be removed on applying integration by parts. Furthermore, if $N\geqslant 100M$, then by the first derivative test the contribution of the expression in (4.8) is

$$\ll T^{1/2}\sum_{M<m\leqslant 2M} d(m)m^{-3/4}\sum_{N<n\leqslant 2N} d(n)n^{-3/4}\ll T\log^2 T, \tag{4.9}$$

and the same bound as in (4.9) holds when $M \geqslant 100N$. These sums in total make a contribution which is $\ll T\log^4 T$.

There remains the case when $N/100 < M < 100N$. Then we use the Cauchy-Schwarz inequality for integrals. The contribution is

$$
\begin{aligned}
&\ll \Biggl\{\int_T^{2T}\Bigl|\sum_{N<n\leqslant N'}\frac{d(n)}{n^{1/4}}e^{4\pi i\sqrt{nt}}\Bigr|^2\,\mathrm{d}t \\
&\times \int_T^{2T}\Bigl|\sum_{M<m\leqslant M'}\frac{(-1)^m d(m)}{m^{3/4}}e(t,m)e^{if(t,m)}\Bigr|^2\,\mathrm{d}t\Biggr\}^{1/2}.
\end{aligned}
\tag{4.10}
$$

Here the first integral is estimated as in (4.7), more precisely by

$$
O(TN^{1/2}\log^3 T + T^{1/2}N\log^3 T).
$$

The second integral is, by the first derivative test and Lemma 7,

$$
\ll T\sum_{m\geqslant M}\frac{d^2(m)}{m^{3/2}} + \sum_{M<k\neq m\leqslant M'}\frac{d(k)d(m)e(t,k)e(t,m)}{(km)^{3/4}}\max_{t\in[T,2T]}\frac{1}{|f'(t,m)-f'(t,k)|}.
$$

We have

$$
f'(t,\ell) = \frac{\partial f(t,\ell)}{\partial t} = 2\mathrm{ar}\sinh\sqrt{\frac{\pi\ell}{2t}},
$$

so that by the mean value theorem we obtain

$$
|f'(t,m)-f'(t,k)| \asymp \frac{|\sqrt{k}-\sqrt{m}|}{\sqrt{T}} \qquad (k\neq m,\ T\leqslant t\leqslant 2T).
$$

Hence the last expression above is

$$
\ll TM^{-1/2}\log^3 T + T^{1/2}\log^4 T.
$$

It is seen then, since $M \asymp N$, that the expression in (4.10) is

$$
\begin{aligned}
&\ll \Bigl((TM^{1/2}\log^3 T + T^{1/2}M\log^3 T)(TM^{-1/2}\log^3 T + T^{1/2}\log^4 T)\Bigr)^{1/2} \\
&\ll (T^2\log^6 T + T^{3/2}M^{1/2}\log^7 T)^{1/2}.
\end{aligned}
$$

Taking $M = T2^{-j}$ and summing over j we obtain that the contribution of $J_1(T)$ is $O(T\log^4 T)$, since $M \asymp N$ in the relevant cases. This gives

$$
\int_T^{2T}\Delta(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \ll T\log^4 T,
$$

and thus completes the proof of Theorem 1.

5. The proof of Theorem 2

Like in the proof of Theorem 1 it suffices to prove the result for the integral over $[T,\,2T]$, where $T\,(\geqslant 10)$ is large. Henceforth let

$$T \leqslant t \leqslant 2T, \quad T^{1/2} \ll y = y(T) \ll T, \tag{5.1}$$

where y will be determined later. Write

$$\Delta(t) = \Delta_1(t,y) + \Delta_2(t,y), \tag{5.2}$$

where

$$\Delta_1(t,y) := \frac{t^{1/4}}{\sqrt{2}\pi}\sum_{n\leqslant y}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4}), \tag{5.3}$$

and by Lemma 4 (with $N = y$)

$$\Delta_2(t,y) \ll_\varepsilon T^{1/2+\varepsilon}y^{-1/2} \quad (\ll_\varepsilon T^{1/4+\varepsilon}). \tag{5.4}$$

Then we have

$$\begin{aligned} &\int_T^{2T} \Delta^k(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\\ &= \int_T^{2T} \Big(\Delta_1(t,y)+\Delta_2(t,y)\Big)^k|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\\ &= \int_1 + O\left(\int_2 + \int_3\right), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \int_1 &:= \int_T^{2T} \Delta_1^k(t,y)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t,\\ \int_2 &:= \int_T^{2T} |\Delta_1^{k-1}(t,y)\Delta_2(t,y)||\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t,\\ \int_3 &:= \int_T^{2T} |\Delta_2(t,y)|^k|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t. \end{aligned}$$

In order to estimate $\int_3$ we need (5.4) and

$$\int_T^{2T} |\Delta_2(t,y)|^2\,\mathrm{d}t \ll_\varepsilon T^{3/2+\varepsilon}y^{-1/2}, \tag{5.6}$$

which follows as in the proof of (4.3). From (5.4), (5.6), the fourth power moment of $\zeta(\frac12+it)$ and the Cauchy-Schwarz inequality (Lemma 10) we obtain

$$
\begin{aligned}
\int_3 &\ll_\varepsilon \left(\frac{T^{1/2+\varepsilon}}{y^{1/2}}\right)^{k-1}\int_T^{2T}|\Delta_2(t,y)||\zeta(\tfrac12+it)|^2\,\mathrm{d}t \\
&\ll_\varepsilon \left(\frac{T^{1/2+\varepsilon}}{y^{1/2}}\right)^{k-1}\left(\int_T^{2T}|\Delta_2(t,y)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_T^{2T}|\zeta(\tfrac12+it)|^4\,\mathrm{d}t\right)^{1/2} \\
&\ll_\varepsilon T^{\frac{2k+3}{4}+\varepsilon}y^{-\frac{2k-1}{4}}.
\end{aligned}
\tag{5.7}
$$

Now we evaluate $\int_1$. We write (2.10) as

$$|\zeta(\tfrac12+it)|^2 = \log t + C + E'(t),$$

where henceforth we set $C = 2\gamma - \log 2\pi$ for brevity. Therefore we have

$$
\begin{aligned}
\int_1 &= \int_T^{2T}\Delta_1^k(t,y)(\log t + C)\,\mathrm{d}t + \int_T^{2T}\Delta_1^k(t,y)E'(t)\,\mathrm{d}t \\
&= \int_{11} + \int_{12},
\end{aligned}
\tag{5.8}
$$

say. We bound first $\int_{12}$. Using integration by parts and Lemma 1 we obtain

$$
\begin{aligned}
\int_{12} &= \Delta_1^k(t,y)E(t)\Big|_T^{2T} - k\int_T^{2T}\Delta_1^{k-1}(t,y)\Delta_1'(t,y)E(t)\,\mathrm{d}t \\
&\ll_\varepsilon T^{(k+1)\theta+\varepsilon} + \left|\int_{12}^*\right|,
\end{aligned}
\tag{5.9}
$$

say, where

$$\int_{12}^* := \int_T^{2T}\Delta_1^{k-1}(t,y)\Delta_1'(t,y)E(t)\,\mathrm{d}t.$$

In order to bound $\int_{12}^*$, we need upper bounds for the second and the fourth moment of $\Delta_1'(t,y)$. It is easily seen that

$$
\begin{aligned}
\Delta_1'(t,y) &= \frac{t^{-3/4}}{4\sqrt{2}\pi}\sum_{n\leqslant y}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{nt}-\tfrac{\pi}{4}) \\
&\quad - \sqrt{2}t^{-1/4}\sum_{n\leqslant y}\frac{d(n)}{n^{1/4}}\sin(4\pi\sqrt{nt}-\tfrac{\pi}{4}) \\
&\ll t^{-1}|\Delta_1(t,y)| + t^{-1/4}\left|\sum_{n\leqslant y}\frac{d(n)}{n^{1/4}}e(2\sqrt{nt})\right|.
\end{aligned}
\tag{5.10}
$$

Since $\Delta_2(t,y) \ll T^{1/4+\varepsilon}$, by (5.4), it follows that

$$\Delta_1(t,y) \ll_\varepsilon |\Delta(t)| + T^{1/4+\varepsilon}.$$

Thus by Lemma 2 we have, for any $0 \leqslant A \leqslant 11$,

$$\begin{aligned}\int_T^{2T} |\Delta_1(t,y)|^A \,\mathrm{d}t &\ll_\varepsilon \int_T^{2T} |\Delta(t)|^A \,\mathrm{d}t + T^{1+A/4+\varepsilon} \\ &\ll_\varepsilon T^{1+M(A)+\varepsilon},\end{aligned} \tag{5.11}$$

where $M(A)$ is as in (3.4) of Lemma 2. For the mean square of $\Delta_1'(t,y)$ we have, by (5.10),

(5.12)

$$\begin{aligned}&\int_T^{2T} |\Delta_1'(t,y)|^2 \,\mathrm{d}t \ll \int_T^{2T} t^{-2}|\Delta_1(t,y)|^2 \,\mathrm{d}t \\ &+ \int_T^{2T} t^{-1/2}\left|\sum_{n\leqslant y} \frac{d(n)}{n^{1/4}} e(2\sqrt{nt})\right|^2 \,\mathrm{d}t \\ &\ll T^{-1/2} + T^{-1/2}\sum_{m,n\leqslant y} \frac{d(m)d(n)}{(mn)^{1/4}}\left|\int_T^{2T} e(2(\sqrt{m}-\sqrt{n})\sqrt{t}\,)\,\mathrm{d}t\right| \\ &\ll T^{1/2}\sum_{n\leqslant y}\frac{d^2(n)}{n^{1/2}} + T^{-1/2}\sum_{m\neq n\leqslant y} \frac{d(m)d(n)}{(mn)^{1/4}}\left|\int_T^{2T} e(2(\sqrt{m}-\sqrt{n})\sqrt{t}\,)\,\mathrm{d}t\right| \\ &\ll T^{1/2}\sum_{n\leq y}\frac{d^2(n)}{n^{1/2}} + \sum_{m\neq n\leqslant y}\frac{d(m)d(n)}{(mn)^{1/4}|\sqrt{m}-\sqrt{n}|} \\ &\ll (yT)^{1/2}\log^3 T,\end{aligned}$$

where we used the first derivative test and Lemma 8.

For the fourth moment of $\Delta_1'(t,y)$ we have, by (5.10), that

$$\begin{aligned}&\int_T^{2T} |\Delta_1'(t,y)|^4 dt \ll \int_T^{2T} t^{-4}|\Delta_1(t,y)|^4 \,\mathrm{d}t \\ &+ \int_T^{2T} t^{-1}\left|\sum_{n\leq y} \frac{d(n)}{n^{1/4}} e(2\sqrt{nt})\right|^4 \,\mathrm{d}t \\ &\ll_\varepsilon T^{-2} + T^{-1+\varepsilon}\int_T^{2T}\left|\sum_{n\sim N} \frac{d(n)}{n^{1/4}} e(2\sqrt{nt})\right|^4 \,\mathrm{d}t\end{aligned}$$

for some $1 \ll N \ll y$. Therefore we have ($a \sim b$ means that $b \leqslant a \leqslant b' \leqslant 2b$)

$$
\begin{aligned}
\int_T^{2T} |\Delta_1'(t,y)|^4 \,\mathrm{d}t &\ll_\varepsilon T^{-2} + T^{-1+\varepsilon} \int_T^{2T} \left| \sum_{n\sim N} \frac{d(n)}{n^{1/4}} e(2\sqrt{nt}) \right|^4 \mathrm{d}t \\
&\ll_\varepsilon T^{-2} + T^{-1+\varepsilon} \sum_{n_1,n_2,n_3,n_4\sim N} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{1/4}} \\
&\times \left| \int_T^{2T} e\Big(2(\sqrt{n_1}+\sqrt{n_2}-\sqrt{n_3}-\sqrt{n_4})t\Big)\,\mathrm{d}t \right| \\
&\ll_\varepsilon \frac{T^{-1+\varepsilon}}{N} \sum_{n_1,n_2,n_3,n_4\sim N} \min\left(T, \frac{\sqrt{T}}{|\Omega|}\right),
\end{aligned}
\tag{5.13}
$$

Here we used trivial estimation and the first derivative test, and we set

$$
\Omega \ := \ \sqrt{n_1}+\sqrt{n_2}-\sqrt{n_3}-\sqrt{n_4}.
$$

Note that $\min\big(T, \sqrt{T}/|\Omega|\big) = T$ if $|\Omega| \leqslant T^{-1/2}$. In this case the contribution to the last sum in (5.13) is, by (3.10) of Lemma 6,

$$
\begin{aligned}
&\ll_\varepsilon \frac{T^{-1+\varepsilon}}{N} T(T^{-1/2}N^{7/2}+N^2) \ll_\varepsilon (T^{-1/2}N^{5/2}+N)T^\varepsilon \\
&\ll_\varepsilon (T^{-1/2}y^{5/2}+y)T^\varepsilon \ll_\varepsilon T^{-1/2+\varepsilon}y^{5/2},
\end{aligned}
$$

on noting that $y \gg T^{1/2}$. If $|\Omega| > T^{-1/2}$, then $\min\big(T, \sqrt{T}/|\Omega|\big) = \sqrt{T}/|\Omega|$. By Lemma 6 again, the contribution is

$$
\begin{aligned}
&\ll_\varepsilon \max_{T^{-1/2}<\eta\ll N^{1/2}} \frac{T^{-1/2+\varepsilon}}{N\eta} \sum_{|\Omega|\sim\eta} 1 \\
&\ll_\varepsilon \max_{T^{-1/2}<\eta\ll N^{1/2}} \frac{T^{-1/2+\varepsilon}}{N\eta} (\eta N^{7/2}+N^2) \\
&\ll_\varepsilon (T^{-1/2}y^{5/2}+y)T^\varepsilon \ll_\varepsilon T^{-1/2+\varepsilon}y^{5/2}.
\end{aligned}
$$

Inserting the above two estimates into (5.13) we obtain

$$
\int_T^{2T} |\Delta_1'(t,y)|^4 \,\mathrm{d}t \ \ll_\varepsilon \ T^{-1/2+\varepsilon}y^{5/2}. \tag{5.14}
$$

Now we bound $\int_{12}^*$. When $k = 2,3,4$, by Hölder's inequality, (5.11), (5.12) and

Lemma 1 we have

$$
\begin{aligned}
\int_{12}^{*} &= \int_T^{2T} \Delta^{k-1}(t,y)\Delta'(t,y)E(t)\,\mathrm{d}t \\
&\ll \left(\int_T^{2T} |\Delta_1'(t,y)|^2 dt\right)^{\frac12} \left(\int_T^{2T} |\Delta_1(t,y)|^{2k}\,\mathrm{d}t\right)^{\frac{k-1}{2k}} \\
&\times \left(\int_T^{2T} |E(t)|^{2k}\,\mathrm{d}t\right)^{\frac{1}{2k}} \ll_\varepsilon T^{\frac{k}{4}+\frac34+\varepsilon} y^{\frac14}.
\end{aligned}
\tag{5.15}
$$

When $k = 5, 6, 7, 8$, by Hölder's inequality again, (5.11), (5.14) and Lemma 1 we have

$$
\begin{aligned}
\int_{12}^{*} &\ll \left(\int_T^{2T} |\Delta_1'(t,y)|^4\,\mathrm{d}t\right)^{\frac14} \times \left(\int_T^{2T} |\Delta_1(t,y)|^{\frac{4k}{3}}\,\mathrm{d}t\right)^{\frac{3k-3}{4k}} \\
&\times \left(\int_T^{2T} |E(t)|^{\frac{4k}{3}}\,\mathrm{d}t\right)^{\frac{3}{4k}} \ll_\varepsilon T^{\frac58+\frac34 M(\frac{4k}{3})+\varepsilon} y^{\frac58}.
\end{aligned}
\tag{5.16}
$$

In (3.4) we have $M(A) = A/4$ for $A \leqslant 262/27 = 9,\overline{703}$, and $M(A) = 131(A-2)/416$ for $262/27 \leqslant A \leqslant 11$. Thus by Lemma 2, inserting (5.15) and (5.16) into (5.9) we obtain

$$
\int_{12} \ll_\varepsilon T^{(k+1)\theta+\varepsilon} + \begin{cases} T^{\frac{k}{4}+\frac34+\varepsilon} y^{\frac14}, \text{ if } k = 2,3,4, \\ T^{\frac58+\frac{k}{4}+\varepsilon} y^{\frac58}, \text{ if } k = 5,6,7, \\ T^{\frac{171}{64}+\varepsilon} y^{\frac58}, \text{ if } k = 8. \end{cases}
\tag{5.17}
$$

Now we evaluate $\int_{11}$ (see (5.8)). Using $\Delta_1(t,y) = \Delta(t) - \Delta_2(t,y)$, we have

$$
\int_{11} = \int_T^{2T} \Delta_1^k(t,y)(\log t + C)\,\mathrm{d}t = \int_4 + O\left\{\int_5 + \int_6\right\},
$$

say, where

$$
\begin{aligned}
\int_4 &= \int_T^{2T} \Delta^k(t)(\log t + C)\,\mathrm{d}t, \\
\int_5 &= \int_T^{2T} |\Delta^{k-1}(t)\Delta_2(t,y)|(\log t + C)\,\mathrm{d}t, \\
\int_6 &= \int_T^{2T} |\Delta_2^k(t,y)|(\log t + C)\,\mathrm{d}t.
\end{aligned}
\tag{5.18}
$$

From (5.4) and (5.6) we infer that

$$\int_6 \ll_\varepsilon \left(\frac{T^{1/2+\varepsilon}}{y^{1/2}}\right)^{k-2} \int_T^{2T} |\Delta_2(t,y)|^2 \,\mathrm{d}t \ll_\varepsilon T^{\frac{k+1}{2}+\varepsilon} y^{-\frac{k-1}{2}}.$$

By Cauchy's inequality, (5.6) and Lemma 2 we have, if $k = 2,3,4,5$, that

$$\begin{aligned}\int_5 &\ll \log T \left(\int_T^{2T} |\Delta_2(t,y)|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_T^{2T} |\Delta(t)|^{2k-2} dt\right)^{1/2} \\ &\ll_\varepsilon T^{1+k/4+\varepsilon} y^{-1/4}.\end{aligned} \tag{5.19}$$

Similarly we obtain by Hölder's inequality, when $k = 6,7,8$,

$$\begin{aligned}\int_5 &\ll \log T \left(\int_T^{2T} |\Delta_2(t,y)|^4 \,\mathrm{d}t\right)^{1/4} \left(\int_T^{2T} |\Delta(t)|^{4(k-1)/3} \,\mathrm{d}t\right)^{3/4} \\ &\ll_\varepsilon T^\varepsilon \left(\frac{T}{y}\int_T^{2T} |\Delta_2(t,y)|^2 \,\mathrm{d}t\right)^{1/4} \left(\int_T^{2T} |\Delta(t)|^{4(k-1)/3} \,\mathrm{d}t\right)^{3/4} \\ &\ll_\varepsilon T^\varepsilon \left(\frac{T^{5/2}}{y^{3/2}}\right)^{1/4} \left(T^{1+(k-1)/3}\right)^{3/4} \\ &= T^{9/8+k/4+\varepsilon} y^{-3/8},\end{aligned} \tag{5.20}$$

where we used $M(4(k-1)/3) = (k-1)/3$ by Lemma 2, since $4(k-1)/3 \leqslant 28/3$. Namely, for $k \leqslant 8$ we have $(4k-4)/3 \leqslant 28/3$, and by (3.4) with $\theta = 131/416$ one obtains $M(A) = A/4$ for $A \leqslant 262/27 = 9.70370\ldots$, while $28/3 = 9.3333\ldots$.

Combining (5.19) and (5.20) with the above estimate for $\int_6$, we obtain

$$\int_5 + \int_6 \ll_\varepsilon \begin{cases} T^{1+k/4+\varepsilon} y^{-1/4}, \text{ if } k = 2,3,4,5, \\ T^{9/8+k/4+\varepsilon} y^{-3/8}, \text{ if } k = 6,7,8. \end{cases} \tag{5.21}$$

From (5.1), (5.17) and (5.21) we have

$$\int_1 = \int_T^{2T} \Delta^k(t)(\log t + C)\,\mathrm{d}t + O_\varepsilon\Big\{G_{k1}(T,y)T^\varepsilon + G_{k2}(T,y)T^\varepsilon\Big\}, \tag{5.22}$$

say, where we have set

$$G_{k1}(T,y) := \begin{cases} T^{1+k/4+\varepsilon} y^{-1/4}, \text{ if } k = 2,3,4,5, \\ T^{9/8+k/4+\varepsilon} y^{-3/8}, \text{ if } k = 6,7,8, \end{cases} \tag{5.23}$$

and

$$G_{k2}(T,y) := T^{(k+1)\theta+\varepsilon} + \begin{cases} T^{\frac{k}{4}+\frac{3}{4}+\varepsilon}y^{\frac{1}{4}}, \text{ if } k=2,3,4,\\ T^{\frac{5}{8}+\frac{k}{4}+\varepsilon}y^{\frac{5}{8}}, \text{ if } k=5,6,7,\\ T^{\frac{171}{64}+\varepsilon}y^{\frac{5}{8}}, \text{ if } k=8. \end{cases} \tag{5.24}$$

Now we estimate $\int_2$ (see (5.5)). Taking $k=2,4,6$ in the estimate (5.22)–(5.24) we obtain

$$\int_T^{2T} |\Delta_1(t,y)|^k|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{1+k/4+\varepsilon} \quad (k=2,4,6;\ T^{1/2}\ll y\ll T^{3/5}). \tag{5.25}$$

Similarly, taking $k=8$ in (5.22)–(5.24) we obtain

$$\int_T^{2T} |\Delta_1(t,y)|^8|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{3+\varepsilon} \quad (T^{1/2}\ll y\ll T^{21/40}), \tag{5.26}$$

which combined with Hölder's inequality implies, for any $2\leqslant A\leqslant 8$, that

$$\begin{aligned}&\int_T^{2T} |\Delta_1(t,y)|^A|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\\ &\ll \left(\int_T^{2T} |\Delta_1(t,y)|^8|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\right)^{\frac{A}{8}} \left(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\right)^{1-\frac{A}{8}}\\ &\ll_\varepsilon T^{1+A/4+\varepsilon}\end{aligned} \tag{5.27}$$

if $T^{1/2}\ll y\ll T^{21/40}$.

When $k=2,3,4$, from (5.7) with $k=2$, (5.25) and the Cauchy-Schwarz inequality we obtain, for $T^{1/2}\ll y\ll T^{3/5}$, that

$$\begin{aligned}\int_2 &\ll \left(\int_T^{2T} |\Delta_2(t,y)|^2|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\right)^{1/2}\\ &\times\left(\int_T^{2T} |\Delta_1(t,y)|^{2k-2}|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\right)^{1/2}\\ &\ll_\varepsilon T^{\frac{9}{8}+\frac{k}{4}+\varepsilon}y^{-\frac{3}{8}}.\end{aligned}$$

When $k=5,6,7$, from (5.7) with $k=4$, (5.25) and Hölder's inequality we have,

for $T^{1/2} \ll y \ll T^{21/40}$, that

$$\begin{aligned}\int_2 &\ll \left(\int_T^{2T} |\Delta_2(t,y)|^4 |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^{1/4}\\ &\times \left(\int_T^{2T} |\Delta_1(t,y)|^{4(k-1)/3} |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^{3/4}\\ &\ll_\varepsilon T^{\frac{19}{16}+\frac{k}{4}+\varepsilon} y^{-\frac{7}{16}}.\end{aligned}$$

When $k=8$, from (5.7) with with $k=8$, (5.27) with $A=8$ and Hölder's inequality we have, for $T^{1/2} \ll y \ll T^{21/40}$,

$$\begin{aligned}\int_2 &\ll \left(\int_T^{2T} |\Delta_2(t,y)|^8 |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^{1/8}\\ &\times \left(\int_T^{2T} |\Delta_1(t,y)|^8 |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^{7/8}\\ &\ll_\varepsilon T^{\frac{103}{32}+\varepsilon} y^{-\frac{15}{32}}.\end{aligned}$$

By combining the above three estimates it follows that

$$\int_2 \ll G_{k3}(T,y) := \begin{cases} T^{\frac98+\frac{k}{4}+\varepsilon} y^{-\frac38}, \text{ when } k=2,3,4,\\ T^{\frac{19}{16}+\frac{k}{4}+\varepsilon} y^{-\frac{7}{16}}, \text{ when } k=5,6,7,\\ T^{\frac{103}{32}+\varepsilon} y^{-\frac{15}{32}}, \text{ when } k=8.\end{cases} \tag{5.28}$$

From (5.5), (5.7), (5.22) and (5.28) we have

$$\begin{aligned}\int_T^{2T} \Delta^k(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t &= \int_T^{2T} \Delta^k(t)(\log t + C)\,\mathrm{d}t\\ &+ O_\varepsilon\left(\sum_{j=1}^{3} G_{kj}(T,y)T^\varepsilon + T^{\frac{2k+3}{4}+\varepsilon} y^{-\frac{2k-1}{4}}\right),\end{aligned} \tag{5.29}$$

where $G_{kj}(T,y)$ $(j=1,2,3)$ was defined in (5.23), (5.24) and (5.28), respectively. It is easy to see that

$$\sum_{j=1}^{3} G_{kj}(T,y) \ll \begin{cases} T^{\frac98+\frac{k}{4}} y^{-\frac38} + T^{\frac34+\frac{k}{4}} y^{\frac14} + T^{(k+1)\theta}, \text{ when } k=2,3,4,\\ T^{\frac{19}{16}+\frac{k}{4}} y^{-\frac{7}{16}} + T^{\frac58+\frac{k}{4}} y^{\frac58} + T^{(k+1)\theta}, \text{ when } k=5,6,7,\\ T^{\frac{103}{32}} y^{-\frac{15}{32}} + T^{\frac{171}{64}} y^{\frac58} + T^{9\theta}, \text{ when } k=8.\end{cases} \tag{5.30}$$

Now taking

$$y = \begin{cases} T^{\frac35}, \text{ when } k = 2,3,4, \\ T^{\frac{21}{40}}, \text{ when } k = 5,6,7, \\ T^{\frac12}, \text{ when } k = 8, \end{cases}$$

we obtain

$$\sum_{j=1}^{3} G_{kj}(T,y) + T^{\frac{2k+3}{4}} y^{-\frac{2k-1}{4}} \ll T^{1+\frac{k}{4}-\eta_k^{**}}, \tag{5.31}$$

where

$$\eta_k^{**} := \begin{cases} 1/10, \text{ when } k = 2,3,4, \\ 27/640, \text{ when } k = 5,6,7, \\ 1/64, \text{ when } k = 8. \end{cases} \tag{5.32}$$

In the case when $k = 2,3,4,8$ we equalize the terms containing y in (5.30), and $(k+1)\theta < 1+k/4-\eta_k^{**}$ holds. In the case when $k = 5,6,7$, note that $T^{\frac{19}{16}+\frac{k}{4}} y^{-\frac{7}{16}} \geqslant T^{\frac58+\frac{k}{4}} y^{\frac58}$ for $T^{1/2} \leqslant y \leqslant T^{9/17}$ but as $y \ll T^{21/40}$ has to hold and $21/40 < 9/17$, we take $y = T^{\frac{21}{40}}$ to obtain (5.31) in this case as well.

From (5.29)-(5.32) we obtain

$$\int_T^{2T} \Delta^k(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \int_T^{2T} \Delta^k(t)(\log t + C)\,\mathrm{d}t + O_\varepsilon\left(T^{1+\frac{k}{4}-\eta_k^{**}+\varepsilon}\right),$$

which implies that

$$\int_1^{T} \Delta^k(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \int_1^{T} \Delta^k(t)(\log t + C)dt + O_\varepsilon\left(T^{1+\frac{k}{4}-\eta_k^{**}+\varepsilon}\right). \tag{5.33}$$

From (5.33), (2.1) and integration by parts we have ($\eta_k^* \equiv c(k)$)

$$\begin{aligned}
&\int_1^{T} \Delta^k(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \\
&= C_k\Big(1+\frac{k}{4}\Big)\int_1^T t^{\frac{k}{4}}(\log t + C)\,\mathrm{d}t + O_\varepsilon\left(T^{1+\frac{k}{4}-\eta_k^*+\varepsilon} + T^{1+\frac{k}{4}-\eta_k^{**}+\varepsilon}\right) \\
&= C_k T^{1+\frac{k}{4}}\left(\log T + C - \frac{4}{k+4}\right) + O_\varepsilon\left(T^{1+\frac{k}{4}-\eta_k^*+\varepsilon} + T^{1+\frac{k}{4}-\eta_k^{**}+\varepsilon}\right) \\
&= c_1(k)T^{1+\frac{k}{4}}\log T + c_2(k)T^{1+\frac{k}{4}} + O_\varepsilon\left(T^{1+\frac{k}{4}-\eta_k+\varepsilon}\right),
\end{aligned}$$

where

$$\begin{aligned} c_1(k) &= C_k, \quad c_2(k) = C_k\left(C - \frac{4}{k+4}\right), \\ \eta_k &= \min(\eta_k^*,\, \eta_k^{**}) \quad (2 \leqslant k \leqslant 8), \end{aligned}$$

so that

$$\eta_2 = \eta_3 = \eta_4 = 1/10,\ \eta_5 = 3/80,\ \eta_6 = 35/4742,\ \eta_7 = 17/6312,\ \eta_8 = 8/9433.$$

This ends the proof of Theorem 2.

6. Proof of Theorem 3

We retain the notation of Section 5. The main task is to evaluate

$$\int_{11} := \int_T^{2T} \Delta_1^2(t,y)(\log t + C)\,\mathrm{d}t$$

and to bound

$$\int_{12}^* := \int_T^{2T} \Delta_1(t,y)\Delta_1'(t,y)E(t)\,\mathrm{d}t.$$

By using (5.3) we have

$$\begin{aligned} \int_{11} &= \frac{1}{2\pi^2}\sum_{m,n\leqslant y}\frac{d(m)d(n)}{(mn)^{3/4}} \\ &\quad\times \int_T^{2T} t^{1/2}(\log t + C)\cos(4\pi\sqrt{mt} - \pi/4)\cos(4\pi\sqrt{nt} - \pi/4)\,\mathrm{d}t. \end{aligned} \tag{6.1}$$

We use the identity

$$\cos\alpha\cos\beta = \tfrac12\Big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big)$$

with $\alpha = 4\pi\sqrt{mt} - \pi/4, \beta = 4\pi\sqrt{nt} - \pi/4$. The terms coming from $\cos(\alpha+\beta)$ make, by the first derivative test, a contribution which is $\ll T\log^5 T$. The same bound holds for the terms coming from $\cos(\alpha-\beta)$ when $m \neq n$. Finally, the terms $m = n$ contribute

$$\begin{aligned} &\frac{1}{4\pi^2}\sum_{n\leqslant y} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}(\log t + C)\,\mathrm{d}t \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}(\log t + C)\,\mathrm{d}t + O(T^{3/2}y^{-1/2}\log^4 T). \end{aligned}$$

It follows that

$$\int_{11} = \frac{1}{4\pi^2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}(\log t + C)\,\mathrm{d}t \tag{6.2}$$
$$+ O(T^{3/2}y^{-1/2}\log^4 T) + O(T\log^5 T).$$

Now we estimate $\int_{12}^{*}$. Here we use the method of proof of Theorem 1 and replace $E(t)$ by Lemma 9 (Atkinson's formula with $N = T$). We write

$$\int_{12}^{*} = \int_{121}^{*} + \int_{122}^{*},$$
$$\int_{121}^{*} = \int_T^{2T} \Delta_1(t,y)\Delta_1'(t,y)\sum\nolimits_1(t)\,\mathrm{d}t, \tag{6.3}$$
$$\int_{122}^{*} = \int_T^{2T} \Delta_1(t,y)\Delta_1'(t,y)\Big(\sum\nolimits_2(t) + O(\log^2 t)\Big)\,\mathrm{d}t.$$

By Hölder's inequality we obtain

$$\int_{122}^{*} \ll \left\{\int_T^{2T}\Big(\sum\nolimits_2(t) + O(\log^2 t)\Big)^2\,\mathrm{d}t\right\}^{1/2} \tag{6.4}$$
$$\times\left\{\int_T^{2T}|\Delta_1'(t,y)|^3\,\mathrm{d}t\right\}^{1/3}\left\{\int_T^{2T}|\Delta_1(t,y)|^6\,\mathrm{d}t\right\}^{1/6}$$
$$\ll_\varepsilon T^{11/12+\varepsilon}y^{1/2}.$$

Here we bounded the mean square of $\Sigma_2(t)$ as after (4.6), used the bound

$$\int_T^{2T}|\Delta_1'(t,y)|^3\,\mathrm{d}t \ll_\varepsilon T^\varepsilon y^{3/2},$$

which follows from the Cauchy-Schwarz inequality from (5.12) and (5.14), and (5.11) with $A = 6, M(6) = 3/2$.

Having in mind (5.10), we see that the major contribution to $\int_{121}^{*}$ comes from a multiple of

$$\int_T^{2T}\sum\nolimits_1(t)\sum_{n\leqslant y} d(n)n^{-1/4}\sin(4\pi\sqrt{nt} - \tfrac{\pi}{4}) \tag{6.5}$$
$$\times\sum_{m\leqslant y} d(m)m^{-3/4}\cos(4\pi\sqrt{mt} - \tfrac{\pi}{4})\,\mathrm{d}t.$$

We use the explicit expression for $\sum_1$ given by Lemma 9. By a splitting argument one sees that the integral in (6.5) can be written as $O(\log^3 T)$ integrals of the form

$$I := I(T; L, M, N) = \int_T^{2T} \frac{t^{1/4}}{L^{3/4}M^{3/4}N^{1/4}} \sum_{L<\ell\leqslant 2L} c_1(\ell)e(t,\ell)\cos(f(t,\ell))$$
$$\times \sum_{M<m\leqslant 2M} c_2(m)\cos(4\pi\sqrt{mt}-\pi/4) \sum_{N<n\leqslant 2N} c_3(n)\sin(4\pi\sqrt{nt}-\pi/4)\,\mathrm{d}t,$$

say, where the coefficients c_j satisfy

$$c_1(\ell) \ll d(\ell), \quad c_2(m) \ll d(m), \quad c_3(n) \ll d(n),$$

and the functions $e(t,\ell), f(t,\ell)$ are as in Lemma 9.

We consider separately several cases.

Case 1. $L \geqslant 100\max(M,N)$.

In this case I can be written as a linear combination of integrals

$$I' = \frac{1}{L^{3/4}M^{3/4}N^{1/4}} \sum_{L<\ell\leqslant 2L} c_1(\ell) \sum_{M<m\leqslant 2M} c_2(m) \sum_{N<n\leqslant 2N} c_3(n)$$
$$\times \int_T^{2T} t^{1/4}e(t,\ell)\exp\Bigl(if(t,\ell) \pm 4\pi i\sqrt{mt} \pm 4\pi i\sqrt{nt}\Bigr)\,\mathrm{d}t.$$

Then the derivative of the function in the exponential is $\gg \sqrt{\ell/T}$ and (this is similar to the discussion regarding (4.8) and (4.9)), by the first derivative test, we obtain

$$I \ll T^{3/4}y^{3/4}\log^3 T. \tag{6.6}$$

Case 2. $M \geqslant 100\max(L,N)$.

Case 3. $N \geqslant 100\max(L,M)$.

These cases are analogous to Case 1, and thus the analogue of (6.6) will hold.

Case 4. $N < 100\max(L,M),\ L < 100\max(N,M),\ M < 100\max(L,N)$.

In this case, like in (4.10) in the proof of Theorem 1, we shall use mean value estimates. To this end let

$$U_1(t) := \sum_{L<\ell\leqslant 2L} c_1(\ell)e(t,\ell)\cos\bigl(f(t,\ell)\bigr),$$
$$U_2(t) := \sum_{M<m\leqslant 2M} c_2(m)\cos(4\pi\sqrt{mt}-\pi/4),$$
$$U_3(t) := \sum_{N<n\leqslant 2N} c_3(n)\sin(4\pi\sqrt{nt}-\pi/4).$$

We need the bounds

$$\int_T^{2T} |U_1(t)|^2 \,\mathrm{d}t \;\ll\; TL\log^4 T \tag{6.7}$$

and

$$\int_T^{2T} |U_1(t)|^4 \,\mathrm{d}t \;\ll_\varepsilon\; T^\varepsilon(T^{1/2}L^{7/2} + TL^2). \tag{6.8}$$

Note that (6.7) follows directly by squaring out the integrand and integrating, while (6.8) follows by the use of (3.10) of Lemma 6 with $k = 2$, similarly as in (5.13) in the proof of Theorem 2. We also note that the analogues of (6.7) and (6.8) hold for the corresponding integrals of $U_j(t)$ $(j = 2, 3)$.

If Case 4 holds, then we must have

Case 4.1. $L \ll M, L \ll N, M \asymp N$, or

Case 4.2. $M \ll L, M \ll N, L \asymp N$, or

Case 4.3. $N \ll L, N \ll M, L \asymp M$.

Let us consider first the case 4.1. Using (6.7)–(6.8) and its analogues, and Hölder's inequality, we have

$$\begin{aligned}
I &\ll \frac{T^{1/4}}{L^{3/4}M^{3/4}N^{1/4}}\int_T^{2T} |U_1(t)U_2(t)U_3(t)|\,\mathrm{d}t\\
&\ll \frac{T^{1/4}}{L^{3/4}M^{3/4}N^{1/4}}\Bigl(\int_T^{2T} |U_1(t)|^4\,\mathrm{d}t\Bigr)^{\frac14}\Bigl(\int_T^{2T} |U_2(t)|^4\,\mathrm{d}t\Bigr)^{\frac14}\Bigl(\int_T^{2T} |U_3(t)|^2\,\mathrm{d}t\Bigr)^{\frac12}\\
&\ll_\varepsilon \frac{T^{1/4+\varepsilon}}{L^{3/4}M^{3/4}N^{1/4}}(T^{1/8}L^{7/8} + T^{1/4}L^{1/2})(T^{1/8}M^{7/8} + T^{1/4}M^{1/2})T^{1/2}N^{1/2}\\
&\ll_\varepsilon T^\varepsilon(TL^{1/8}M^{3/8} + T^{9/8}L^{1/8} + T^{9/8}M^{3/8}L^{-1/4} + T^{5/4}L^{-1/4}).
\end{aligned}$$

By using the trivial estimate $U_1(t) \ll L\log L$ we also have, since $M \asymp N$,

$$\begin{aligned}
I &\ll \frac{T^{1/4}L^{1/4}\log L}{M}\int_T^{2T} |U_2(t)U_3(t)|\,\mathrm{d}t\\
&\ll \frac{T^{1/4}L^{1/4}\log L}{M}\left(\int_T^{2T} |U_2(t)|^2\,\mathrm{d}t\int_T^{2T} |U_3(t)|^2\,\mathrm{d}t\right)^{1/2}\\
&\ll T^{5/4}L^{1/4}\log^5 T.
\end{aligned}$$

From the last two estimates for I we infer that, when $L \ll M, L \ll N, M \asymp N$,

(6.9)
$$I \ll_\varepsilon T^\varepsilon \left(TL^{1/8}M^{3/8} + T^{9/8}L^{1/8} + T^{5/4}L^{-1/4} + \min\Bigl(\frac{T^{9/8}M^{3/8}}{L^{1/4}},\, T^{5/4}L^{1/4}\Bigr)\right)$$
$$\ll_\varepsilon T^\varepsilon (Ty^{1/2} + T^{19/16}y^{3/16} + T^{5/4}),$$

since $\min(a,b) \leqslant \sqrt{ab}$ for $a, b > 0$.

In the case 4.2, the argument is the same, only the orders of L and M are changed. Consequently the bound (6.9) will hold again. Finally in the case 4.3 we obtain

(6.10)
$$I \ll \frac{T^{1/4}}{L^{3/4}M^{3/4}N^{1/4}} \Bigl(\int_T^{2T} |U_1(t)|^4 \,\mathrm{d}t\Bigr)^{1/4}$$
$$\times \Bigl(\int_T^{2T} |U_2(t)|^4 \,\mathrm{d}t\Bigr)^{1/4} \Bigl(\int_T^{2T} |U_3(t)|^2 \,\mathrm{d}t\Bigr)^{1/2}$$
$$\ll_\varepsilon \frac{T^{1/4+\varepsilon}}{L^{3/4}M^{3/4}N^{1/4}} T^{1/2}N^{1/2}(T^{1/8}M^{7/8} + T^{1/4}M^{1/2})(T^{1/8}L^{7/8} + T^{1/4}L^{1/2})$$
$$\ll_\varepsilon \frac{T^{3/4+\varepsilon}N^{1/4}}{M^{3/2}}(T^{1/4}M^{7/4} + T^{1/2}M)$$
$$\ll_\varepsilon T^{1+\varepsilon}N^{1/4}M^{1/4} + T^{5/4+\varepsilon}N^{1/4}M^{-1/2} \ll_\varepsilon T^{1+\varepsilon}y^{1/2} + T^{5/4+\varepsilon}.$$

Hence (6.9) and (6.10) yield

$$I \ll_\varepsilon T^\varepsilon (Ty^{1/2} + T^{19/16}y^{3/16} + T^{5/4}).$$

Combining the estimates for I in all four cases we have

(6.11)
$$I \ll_\varepsilon T^\varepsilon (T^{3/4}y^{3/4} + Ty^{1/2} + T^{19/16}y^{3/16} + T^{5/4}) \ll_\varepsilon T^\varepsilon (Ty^{1/2} + T^{19/16}y^{3/16} + T^{5/4}),$$

since $T^{3/4}y^{3/4} \leqslant Ty^{1/2}$. Using (6.11) to bound the expression in (6.5) we obtain

$$\int_{121}^{*} \ll_\varepsilon T^{1+\varepsilon}y^{1/2} + T^{19/16+\varepsilon}y^{3/16} + T^{5/4+\varepsilon}. \tag{6.12}$$

From (5.8), (6.2) and (6.12) we have

$$\int_{12}^{*} \ll_\varepsilon T^{1+\varepsilon}y^{1/2} + T^{19/16+\varepsilon}y^{3/16} + T^{5/4+\varepsilon}, \tag{6.13}$$

and this gives in (5.9) (note that $k=2$)

$$\int_{12} \ll_\varepsilon T^{1+\varepsilon}y^{1/2} + T^{19/16+\varepsilon}y^{3/16} + T^{5/4+\varepsilon}. \tag{6.14}$$

From (5.8), (6.2) and (6.14) it follows that

(6.15)
$$\int_1 = \frac{1}{4\pi^2}\sum_{n=1}^\infty d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}(\log t + C)\,\mathrm{d}t$$
$$+ O_\varepsilon(T^{3/2+\varepsilon}y^{-1/2}) + O_\varepsilon(T^{1+\varepsilon}y^{1/2}) + O_\varepsilon(T^{19/16+\varepsilon}y^{3/16}) + O_\varepsilon(T^{5/4+\varepsilon}).$$

It remains yet to deal with $\int_2$ and $\int_3$ in (5.5) when $k=2$. In this case (5.7) yields

$$\int_3 \ll_\varepsilon T^{7/4+\varepsilon}y^{-3/4}. \tag{6.16}$$

Now we bound $\int_2$. We have

$$\int_T^{2T} |\Delta_2(t,y)|^3\,\mathrm{d}t \ll_\varepsilon T^{1/2+\varepsilon}y^{-1/2}\int_T^{2T} |\Delta_2(t,y)|^2\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon}y^{-1}.$$

We use (5.11) with $A=6$, Lemma 3 and Hölder's inequality to obtain that

$$\begin{aligned}\int_2 &\ll \Bigl(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\Bigr)^{1/2}\Bigl(\int_T^{2T} |\Delta_2(t,y)|^3\,\mathrm{d}t\Bigr)^{1/3}\\ &\quad\times\Bigl(\int_T^{2T} |\Delta_1(t,y)|^6\,\mathrm{d}t\Bigr)^{1/6}\\ &\ll_\varepsilon T^{19/12+\varepsilon}y^{-1/3}.\end{aligned} \tag{6.17}$$

Thus from (6.15)–(6.17) it follows that

$$\begin{aligned}\int_T^{2T} \Delta^2(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t &= \frac{1}{4\pi^2}\sum_{n=1}^\infty d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}(\log t + C)\,\mathrm{d}t\\ &+ O_\varepsilon\Bigl(T^\varepsilon(T^{19/12}y^{-1/3} + Ty^{1/2} + T^{19/16}y^{3/16} + T^{5/4})\Bigr),\end{aligned} \tag{6.18}$$

since $T^{3/2}y^{-1/2} \ll Ty^{1/2}$ for $y \gg T^{1/2}$. Finally, taking

$$y = T^{7/10}$$

it is seen that all the error terms in (6.18) are $\ll_\varepsilon T^{27/20+\varepsilon}$, and we obtain from (6.18)

$$\begin{aligned}&\int_1^T \Delta^2(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\\ &= \frac{1}{4\pi^2}\sum_{n=1}^\infty d^2(n)n^{-3/2}\int_1^T t^{1/2}(\log t + C)\,\mathrm{d}t + O_\varepsilon(T^{27/20+\varepsilon})\\ &= c_1(2)T^{3/2}\log T + c_2(2)T^{3/2} + O_\varepsilon(T^{3/2-3/20+\varepsilon}),\end{aligned}$$

which is the assertion of Theorem 3.

References

[1] F.V. Atkinson, *The mean value of the Riemann zeta-function*, Acta Math. **81**(1949), 353-376.

[2] D.R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problems*, Acta Arith. **60**(1992), 389-415.

[3] M. N. Huxley, *Exponential sums and lattice points III,* Proc. London Math. Soc. **87**(3)(2003), 591–609.

[4] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons, New York, 1985 (2nd ed. Dover, Mineola, New York, 2003).

[5] A. Ivić, *On some integrals involving the mean square formula for the Riemann zeta-function*, Publs. Inst. Math. (Belgrade) **46(60)** (1989), 33-42.

[6] A. Ivić, *Large values of certain number-theoretic error terms*, Acta Arith. **56** (1990), 135-159.

[7] A. Ivić, *Mean values of the Riemann zeta-function*, LN's **82**, Tata Inst. of Fundamental Research, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).

[8] A. Ivić, *On the divisor function and the Riemann zeta-function in short intervals*, The Ramanujan Journal, **19**(2009), 207-224.

[9] A. Ivić, *On some mean value results for the zeta-function and a divisor problem*, to appear in Filomat, preprint available at `arXiv:1406.0604`.

[10] A. Ivić and Y. Motohashi, *On the fourth power moment of the Riemann zeta-function*, J. Number Theory **51**(1995), 16-45.

[11] A. Ivić and P. Sargos, *On the higher power moments of the error term in the divisor problem*, Illinois J. Math. **81**(2007), 353-377.

[12] M. Jutila, *Riemann's zeta-function and the divisor problem*, Arkiv Mat. **21**(1983), 75-96 and II, ibid. **31**(1993), 61-70.

[13] M. Jutila, *On a formula of Atkinson*, Topics in classical number theory, Colloq. Budapest 1981, Vol. I, Colloq. Math. Soc. János Bolyai **34**(1984), 807-823.

[14] K.-L. Kong, *Some mean value theorems for certain error terms in analytic number theory*, Master's Thesis, University of Hong Kong, Hong Kong, 2014, 64pp.

[15] Y.-K. Lau, K.-M. Tsang, *On the mean square formula of the error term in the Dirichlet divisor problem*, Math. Proc. Cambridge Philos. Soc. **146**(2009), no. 2, 277-287.

[16] T. Meurman, *On the mean square of the Riemann zeta-function*, Quart. J. Math. Ser. (2)**38** (1987), 337-343.

[17] Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge University Press, Cambridge, 1997.

[18] K. Ramachandra, *On the mean-value and omega-theorems for the Riemann zeta-function*, LN's **85**, Tata Inst. of Fundamental Research (distr. by Springer Verlag, Berlin etc.), Bombay, 1995.

[19] O. Robert and P. Sargos, *Three-dimensional exponential sums with monomials, J. reine angew. Math.* **591**(2006), 1-20.

[20] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew. Math. **31**(1980), 161-170.

[21] K.-M. Tsang, *Recent progress on the Dirichlet divisor problem and the mean square of the Riemann zeta-function*, Sci. China Math. **53**(2010), no. 9, 2561-2572.

[22] G.F. Voronoï, Sur une fonction transcendante et ses applications à la sommation de quelques séries, Ann. École Normale **21**(3)(1904), 207-268 and ibid. **21**(3)(1904), 459-534.

[23] N. Watt, *A note on the mean square of* $|\zeta(\frac{1}{2}+it)|$, J. London Math. Soc. **82(2)**(2010), 279-294.

[24] W. Zhai, *On higher-power moments of* $\Delta(x)$, Acta Arith. **112**(2004), 367-395; II. ibid. **114**(2004), 35-54; III ibid. **118**(2005), 263-281, and IV, Acta Math. Sinica, Chin. Ser. **49** (2006), 639-646.

[25] W. Zhai, *On higher-power moments of* $E(t)$, Acta Arith. **115**(2004), 329-348.

Aleksandar Ivić, Katedra Matematike RGF-a Universiteta u Beogradu, Đušina 7, 11000 Beograd, Serbia

Wenguang Zhai, Department of Mathematics, China University of Mining and Technology, Beijing 100083, P.R.China

E-mail address: aleksandar.ivic@rgf.bg.ac.rs, zhaiwg@hotmail.com